

Limitations of the Method of Lagrangian Descriptors in Incompressible Flows (with Rebuttal to: Response by Mancho, Wiggins et al.)

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Abstract

The method of Lagrangian descriptors has been already applied in many different contexts, specially in geophysical flows. In this paper we analyze the performance of this methodology in incompressible flows. We demonstrate that barriers to transport are not always coded by singular features of the M function as Mancho, Wiggins and their co-workers conjectured. The techniques presented here are not restricted to incompressible flows. In fact, by our approach, one can infer that the method of Lagrangian descriptors is not useful in most linear systems.

1 Introduction

Analytical treatments of transport processes typically impose that the motion of a particle is given by

$$x'(t) = v(t, x(t)) \quad (1.1)$$

where $v(t, x)$ represents the velocity field. The precise description of the trajectories of (1.1) for any initial condition is usually a very difficult task. By this reason, many researchers have focused their attention on a less ambitious problem -the detection of barriers to transport, that is, trajectories acting as skeleton of patterns, (the obstruction to transport takes place by uniqueness of solution), see [18, 17, 25, 11, 24, 23, 19].

To detect barriers to transport in general flows, Mancho, Wiggins, and their co-workers introduced in several papers [5, 8, 11] the method of Lagrangian descriptors. This methodology has been already applied in many different contexts without assumptions on (1.1). For instance, Mendoza and Mancho in [6]

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described the skeleton of the flow for the Kuroshio current; de la Camara *et al* in [7] provided barriers to transport in the Antarctic polar vortex; or Rempel *et al* in [14] studied the saturation of a nonlinear dynamo. Very recently, Craven and Hernandez used the method of Lagrangian descriptors in the context of thermalized chemical reactions [15]. In that letter they claim: “This approach is applicable to any activated system subject to arbitrary driving and thermal fluctuations”. The reader can consult, for instance, [2]-[11] for additional applications.

The lack of mathematical foundations of the method of Lagrangian descriptors makes this approach problematic in precise situations. In [16], the author presented several counter-examples and a mathematical analysis to explain the no connection with invariant manifolds in dynamical systems. Moreover, I discussed phenomena of “ghost dynamics”, i.e. the detection of invariant manifolds at irrelevant points. Most of real geophysical applications deal with incompressible velocity fields. However the counter-examples for this class of systems given in [16] are very complex and have an aperiodic dependence on time.

The objective of the present paper is to improve our theoretical understanding on the performance of the method of Lagrangian descriptors. We are interested when the dynamical behaviour of the flow is governed by a global saddle point. Roughly speaking, our results basically express three limitations:

- The attraction and repulsion rates must be similar in the whole space.
- Any local condition in (1.1) can be irrelevant in the performance of the method. Specifically, we find two autonomous incompressible flows coinciding in $A = (-1, 1) \times \mathbb{R}$ where the responses of Lagrangian descriptors in each system are completely different in that region.
- The method of the Lagrangian descriptors is very sensible to any small perturbation of (1.1).

It is worth mentioning that the techniques presented here offer the possibility to investigate general classes of flows. For instance, one can infer that the method of Lagrangian descriptors is not useful in most linear systems.

The outline of this note is as follows. In the next section we recall the method of Lagrangian descriptors for the reader’s convenience. In Section 3, we provide families of counter-examples in incompressible flows. Finally, we conclude the manuscript with a discussion.

2 The method of Lagrangian Descriptors

In this section we recall some definitions and the precise statement of the method of Lagrangian descriptors.

Consider

$$z' = F(t, z) \tag{2.2}$$

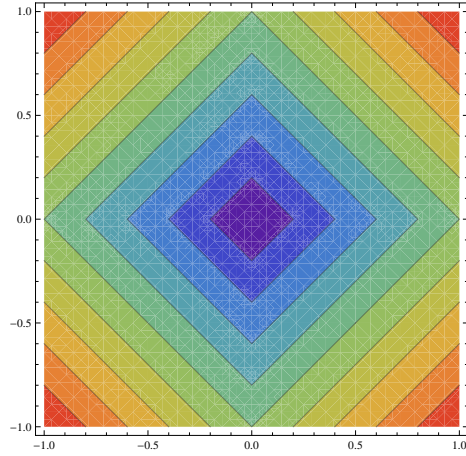


Figure 1: Contour-lines of M in $(-1, 1) \times (-1, 1)$ associated with system (2.3) for $\lambda = 1$ and $\tau = 20$.

where F is of class $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$. We define $M(\tau; z_0)$ as the Euclidean arc length of the solution starting at z_0 along the interval $[-\tau, \tau]$, namely

$$M(\tau; z_0) = \int_{-\tau}^{\tau} \|z'(t; z_0)\| dt.$$

In the previous expression, $z(t; z_0)$ is the solution of (2.2) satisfying that $z(0; z_0) = z_0$. We denote the contour line of M passing through z_* at time $\tau > 0$ as

$$\gamma_{z_*}^{\tau} = \{z \in \mathbb{R}^N : M(\tau; z_*) = M(\tau; z)\}.$$

The method of Lagrangian descriptors says that the invariant manifolds of saddle-points in (2.2) are given by “singular points” (i.e. “non-smooth points”) of the contour lines of M , see [11, 5, 8]. To illustrate this idea, we discuss the example theoretically studied by Mancho, Wiggins and their co-workers in [5], specifically

$$\begin{cases} x' = \lambda x \\ y' = -\lambda y \end{cases} \quad \text{with } \lambda > 0. \quad (2.3)$$

The contour-lines of M associated with (2.3) converge to curves with singular features at the axes, see Figure 1.

Note that the origin is a global saddle point of (2.3) where the x -axis (resp. the y -axis) is its unstable (resp. stable) manifold. A key dynamical property of (2.3) is that the repulsion and attraction rates are exactly the same in the whole phase space. When such a property is violated -even in the simplest way- the method of Lagrangian descriptors can fail.

3 Lagrangian descriptors and Incompressible Flows

In this section we provide several counter-examples to the method of Lagrangian Descriptors in 2D and 3D incompressible flows.

Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ a function of class \mathcal{C}^2 with $f(0) = 0$ and satisfying the following conditions:

C1 $f(x) > 0$ if $x \in (0, \infty)$ and $f(x) < 0$ if $x \in (-\infty, 0)$.

C2 f is bounded.

C3 $f'(0) \geq f'(x) > 0$ for all $x \in \mathbb{R}$.

C4 $f'(x) = k$ for all $x \in [-a, a]$ with $k, a > 0$.

An example of such a function is, for instance,

$$f(x) = \begin{cases} \arctan(x+1) - 1 & \text{if } x \leq -1 \\ x & \text{if } -1 < x < 1 \\ \arctan(x-1) + 1 & \text{if } x \geq 1 \end{cases} \quad (3.4)$$

Theorem 3.1. *Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies **C1-C4**. Then the method of Lagrangian descriptors fails in the Hamiltonian system*

$$\begin{cases} x' = f(x) \\ y' = -yf'(x). \end{cases} \quad (3.5)$$

Before giving the proof of this result, we observe that (3.5) has a global saddle at the origin where the x -axis (resp y -axis) is the un-stable (resp. stable) manifold. Systems (3.5) and (2.3) coincide in $(-1, 1) \times \mathbb{R}$, (when $a = k = 1$). However the response of the method of Lagrangian descriptors in that region is completely different in both systems. In (3.5), on the stable manifold, the attraction has an exponential rate of $f'(0)$ but on the unstable manifold, the repulsion rate is sub-linear.

To prove Theorem 3.1, we need the following lemma.

Lemma 3.1. *Fix a compact set $K \subset [-a, a] \times (0, \infty)$. Then*

$$\lim_{\tau \rightarrow \infty} \frac{M(\tau; x_0, y_0)}{M(\tau; 0, y_0)} = 1$$

for all $(x_0, y_0) \in K$.

Proof. First of all, we note that the solution of (3.5) with initial condition (x_0, y_0) is given by

$$(x(t; x_0, y_0), y(t; x_0, y_0)) = (x(t; x_0), y_0 e^{-\int_0^t f'(x(s; x_0)) ds})$$

where $x(t; x_0)$ is the solution of $x' = f(x)$ with initial condition x_0 . By simple computations and using **C2, C3**,

$$\begin{aligned} M(\tau; 0, y_0) &= y_0(e^{f'(0)\tau} - e^{-f'(0)\tau}), \\ M(\tau; x_0, y_0) &= \int_{-\tau}^{\tau} \sqrt{x'(t; x_0)^2 + y_0^2 e^{-2 \int_0^t f'(x(s; x_0)) ds} f'(x(t; x_0))^2} dt \\ &\leq \int_0^{\tau} \sqrt{f(x(t; x_0))^2 + y_0^2 f'(0)^2} dt + \int_{-\tau}^0 \sqrt{f(x(t; x_0))^2 + y_0^2 e^{-2f'(0)t} f'(0)^2} dt \\ &\leq \int_0^{\tau} \sqrt{m^2 + y_0^2 f'(0)^2} dt + \int_{-\tau}^0 \sqrt{m^2 + y_0^2 e^{-2f'(0)t} f'(0)^2} dt \end{aligned}$$

for all (x_0, y_0) with $y_0 > 0$ where m is a bound of f . Therefore,

$$\lim_{\tau \rightarrow \infty} \frac{M(\tau, x_0, y_0)}{M(\tau, 0, y_0)} \leq 1$$

because of

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \frac{\tau \sqrt{m^2 + y_0^2 f'(0)^2}}{y_0(e^{f'(0)\tau} - e^{-f'(0)\tau})} &= 0, \\ \lim_{\tau \rightarrow \infty} \frac{\int_{-\tau}^0 \sqrt{m^2 + y_0^2 e^{-2f'(0)t} f'(0)^2} dt}{y_0(e^{f'(0)\tau} - e^{-f'(0)\tau})} &= 1. \end{aligned}$$

The second limit is a simple application of the L'Hôpital rule.

By **C1** and **C4**,

$$-a \leq x(t; x_0) \leq a$$

for all $t < 0$ and $-a \leq x_0 \leq a$. Thus, $f'(x(t; x_0)) = f'(0) = k$ for all $t < 0$ and $-a \leq x_0 \leq a$. Now, we conclude that

$$\lim_{\tau \rightarrow \infty} \frac{M(\tau; x_0, y_0)}{M(\tau; 0, y_0)} \geq 1$$

since

$$\begin{aligned} M(\tau; x_0, y_0) &= \int_{-\tau}^{\tau} \sqrt{x'(t; x_0)^2 + y_0^2 e^{-2 \int_0^t f'(x(s; x_0)) ds} f'(x(t; x_0))^2} dt \\ &\geq \int_{-\tau}^0 \sqrt{x'(t; x_0)^2 + y_0^2 e^{-2 \int_0^t f'(x(s; x_0)) ds} f'(x(t; x_0))^2} dt \geq y_0(e^{f'(0)\tau} - 1). \end{aligned}$$

□

Proof of Theorem 3.1. For each $x_0 \in [-a, a]$, we define y_0^τ such that $(x_0, y_0^\tau) \in \gamma_{(0, y_0)}^\tau$. First we observe that for all $\tau > 0$ large enough, y_0^τ exists since

$$M(\tau; x_0, 0) \leq 2m\tau < M(\tau; 0, y_0) = y_0(e^{f'(0)\tau} - e^{-f'(0)\tau})$$

where m is a bound of f and

$$M(\tau; x_0, 2y_0) \geq 2y_0(e^{f'(0)\tau} - 1) \geq M(\tau; 0, y_0).$$

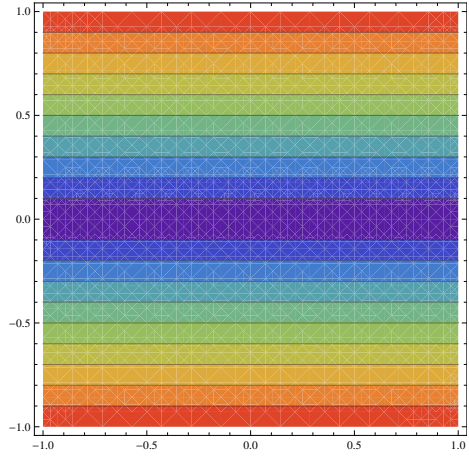


Figure 2: Contour-lines of M in $(-1, 1) \times (-1, 1)$ associated with system (3.5) for (3.4) and $\tau = 20$. The contour lines of M tend to “horizontal lines” in a neighbourhood of the stable manifold.

By Lemma 3.1, if we fix a compact set $K \subset [-a, a] \times (0, \infty)$,

$$M(\tau, x_0, y_0) = M(\tau, 0, y_0) + f(\tau, x_0, y_0)$$

with

$$\lim_{\tau \rightarrow \infty} \frac{f(\tau, x_0, y_0)}{e^{f'(0)\tau} - e^{-f'(0)\tau}} = 0$$

for all $(x_0, y_0) \in K$. Therefore, by using that $M(\tau, 0, y_0) = M(\tau, x_0, y_0^\tau)$, we have that

$$y_0(e^{f'(0)\tau} - e^{-f'(0)\tau}) = y_0^\tau(e^{f'(0)\tau} - e^{-f'(0)\tau}) + f(\tau, x_0, y_0^\tau).$$

From this expression,

$$y_0 = y_0^\tau + \frac{f(\tau, x_0, y_0^\tau)}{e^{f'(0)\tau} - e^{-f'(0)\tau}}$$

and clearly, $\lim_{\tau \rightarrow \infty} y_0^\tau = y_0$. Collecting all the information, we have proved that

$$\gamma_{(0, y_0)}^\tau \cap ([-a, a] \times (0, \infty)) \rightarrow [-a, a] \times \{y_0\}. \quad (3.6)$$

That is, the contour lines in a neighbourhood of the stable manifold tend to horizontal segments. \square

In Figure 2, we illustrate the previous theorem in system (3.5) with function (3.4).

The techniques of the previous theorem allow us to give simple families of counter-examples in linear systems.

Theorem 3.2. *Consider*

$$\begin{cases} x'_1 = \lambda_1 x_1 \\ x'_2 = \lambda_2 x_2 \\ \vdots \\ x'_{N-1} = \lambda_{N-1} x_{N-1} \\ x'_N = -\lambda_N x_N \end{cases} \quad (3.7)$$

with $\lambda_i > 0$ and $\lambda_N > \max\{\lambda_i : i = 1, \dots, N-1\}$. Then the method of Lagrangian Descriptors does not detect the stable manifold of (3.7).

Proof. Denote by $\{e_i : i = 1, \dots, N\}$ the usual basis of \mathbb{R}^N . It is clear that

$$M(\tau; x_i e_i) = x_i (e^{\lambda_i \tau} - e^{-\lambda_i \tau})$$

and

$$M(\tau; x_N e_N) \leq M(\tau; (x_1, \dots, x_N)) \leq M(\tau; x_1 e_1) + \dots + M(\tau; x_N e_N)$$

for all $(x_1, \dots, x_N) \in \mathbb{R}^N$. Therefore, given a compact set $K \subset \mathbb{R}^{N-1} \times (0, \infty)$, we have that

$$\lim_{\tau \rightarrow \infty} \frac{M(\tau; (x_1, \dots, x_N))}{M(\tau; x_N e_N)} = 1$$

for all $(x_1, \dots, x_N) \in K$. By repeating the proof of the previous theorem, we get that

$$\gamma_{x_N e_N}^\tau \cap ([-a, a]^{N-1} \times (0, \infty)) \rightarrow [-a, a]^{N-1} \times \{y_N\} \quad (3.8)$$

with $a > 0$. That is, the contour surfaces of M in a neighbourhood of the stable manifolds tend to smooth surfaces. \square

In Figure 3 we illustrate Theorem 3.2 in \mathbb{R}^3 with $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Note that, in this case, (3.7) is an incompressible flow.

4 Conclusion

In this paper we have analyzed the performance of the method of Lagrangian descriptors in incompressible flows. Our results demonstrate that barriers to transport are not always coded by singular features of the M function as Mancho, Wiggins and their co-workers conjectured in [11, 5, 8]. Moreover, our counter-examples suggest the following pathologies:

- Any local condition can be irrelevant in the performance of this methodology.
- The attraction and repulsion rates of saddle points must be “essentially” the same in the whole space. That limitation can be visualized in a clear manner in

$$\begin{cases} x' = \lambda x \\ y' = -\mu y \end{cases} \quad (4.9)$$

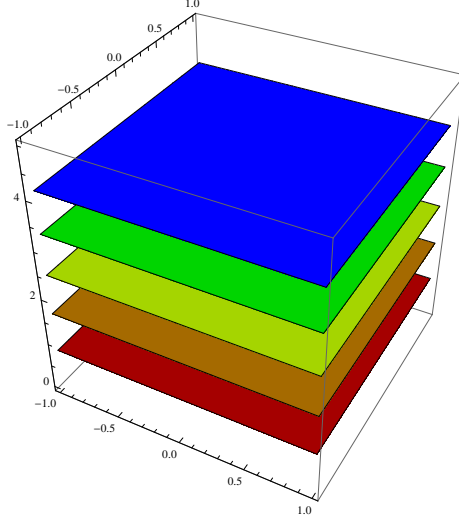


Figure 3: Contour-lines of M in $(-1, 1) \times (-1, 1) \times (0, 5)$ associated with system (3.7) and $\tau = 20$. The contour surfaces of M tend to “horizontal planes” in a neighbourhood of the stable manifold.

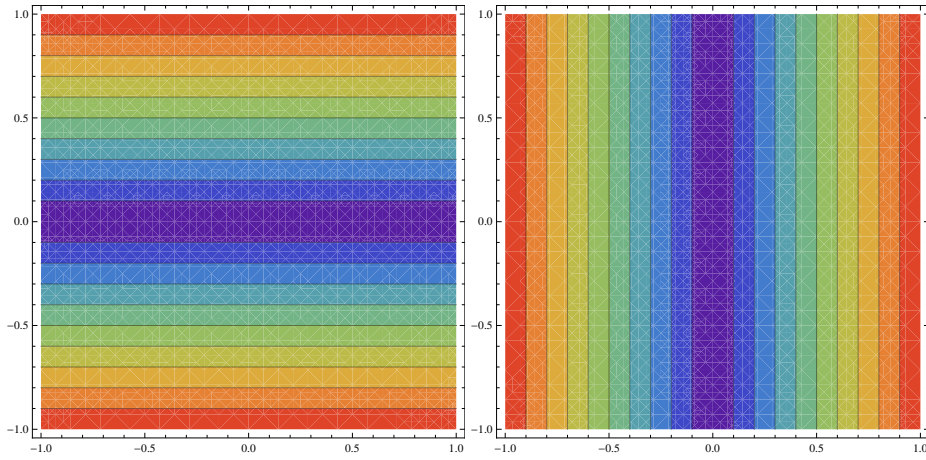


Figure 4: Contour-lines of M in $(-1, 1) \times (-1, 1)$ associated with system (4.9) for $\lambda = 1, \mu = 2$ (Left); $\lambda = 2, \mu = 1$ (Right) and $\tau = 20$. The contour lines of M are “horizontal lines” in a neighbourhood of the stable manifold (Left) and vertical lines in a neighbourhood of the unstable manifold (Right).

with $\lambda \neq \mu > 0$. If $\lambda = \mu$, the method of Lagrangian descriptors works. However, under any simple perturbation of this equality, that claim is not longer true, see Figure 4.

- The counter-examples that we found have simple dynamics. Hence, more pathologies are expected in systems with more complex dynamics.

Numerous Lagrangian diagnostic tools such as finite time Lyapunov exponents or finite size Lyapunov exponents have been applied to infer transport barriers in real applications, see [18, 17, 25, 11, 24]. To the best of my knowledge, a solid mathematical framework based on rigorous theorems, similar to Lagrangian coherent structures results in [17, 20, 22, 21], is completely missing for the method of Lagrangian descriptors. The results of the present paper stress the necessity of such a framework in the applications discussed in [2]-[11].

Finally, we point out that the Poincaré map at time 1 of the systems discussed above are counter-examples to the method of Lagrangian descriptors in discrete systems developed by Lopesino *et al* in [11]. In that paper, they introduced variations of the M function as possible diagnostic tools, specifically

$$M_p(\tau, x_0, y_0) = \int_{-\tau}^{\tau} |x'(t; x_0, y_0)|^p + |y'(t; x_0, y_0)|^p dt$$

with $0 < p < 1$. Palpably, one can see that the role of p in studying systems (4.9) (when $\mu > \lambda$) or (3.5) is to reduce the velocity of attraction to horizontal lines of the contour lines of M_p .

5 Rebuttal to “Response to: Limitations of the method of Lagrangian descriptors by F. Balibrea-Iniesta, J. Curbelo, V.J. García-Garrido, C. Lopesino, A.M. Mancho, C. Mendoza, and S. Wiggins”

This is the second attempt by Mancho, Wiggins, and their co-workers to discard my work in connection with counter-examples to the method of Lagrangian Descriptors. First, they claimed that the proofs in [16] were incorrect (their paper was rejected). Now, they support that I do not understand their method.

According to [5] (page 3534 line -11, see also [2])

In Fig. 1 we show the evolution of the contours of M_1 for $\tau = 0 : 5; 2; 10$. The first thing to observe is that the patterns of the contours displayed depend on τ . For τ small the structure of M_1 is smooth and for increasing τ the contour patterns converge towards a structure that displays the manifolds by means of discontinuity in the derivatives. An analytical argument concerning the convergence time is given in Section 2.

Note that M_1 in [5] is M in this manuscript and Fig 1 coincide in both papers.

Figures presented in [1] show that

$$\frac{\partial M(\tau, 0, y)}{\partial x} = \frac{\partial M(\tau, x, 0)}{\partial y} = 0 \quad (5.10)$$

in the involved systems. Consider $M(\tau, x, y) = \tau^2(x^2 + y^2)$. Obviously, M satisfies (5.10), is of class \mathcal{C}^∞ and their contour lines are circles. Consequently, their figures have no mathematical implications. Again their arguments are wrong.

I have written this letter to avoid confusions in the scientific community. My work was published in Chaos. The journal publishes comments (with external referee process), see

<http://scitation.aip.org/content/aip/journal/chaos>.

Of course, this is the suitable place for [1]. I will provide a detailed response if they publish their paper.

References

- [1] F. Balibrea-Iniesta et al. *Response to: Limitations of the method of Lagrangian Descriptors in incompressible flows*, arXiv:1602.04243.
- [2] C. MENDOZA, A. M. MANCHO, AND S. WIGGINS, *Lagrangian Descriptors and the Assesment of the Predictive Capacity of Oceanic Data Sets*, Nonlinear Processes in Geophysics 21 (2014), 485–501.
- [3] S. WIGGINS AND A. M. MANCHO, *Barriers to tranport in aperiodically time-dependent two dimensional velocity fields: Nekhoroshev's Theorem and 'Nearly Invariant' Tori*, Nonlinear Processes in Geophysics 21 (2014), 165–185.
- [4] A. DE LA CÁMARA, R. MECHOSO, A. M. MANCHO, E. SERRANO, AND K. IDE, *Quasi-horizontal transport within the Antarctic polar night vortex: Rossby wave breaking evidence and Lagrangian structures*, Journal of the Atmospheric Sciences 70 (2013) 2982–3001.
- [5] A. M. MANCHO, S. WIGGINS, J. CURBELO AND C. MENDOZA, *Lagrangian Descriptors: A Method for Revealing Phase Space Structures of General Time Dependent Dynamical Systems*, Communications in Nonlinear Science and Numerical Simulation 18 (2013), 3530–3557.
- [6] C. MENDOZA AND A. M. MANCHO, *The Lagrangian description of aperiodic flows: a case study of the Kuroshio Current*, Nonlinear Processes in Geophysics 19 (2012), 449–472.

- [7] A. DE LA CÁMARA, A. M. MANCHO, K. IDE, E. SERRANO, AND C.R. MECHOSO, *Routes of transport across the Antarctic polar vortex in the southern spring*, Journal of the Atmospheric Sciences 69 (2012) 753–767.
- [8] C. MENDOZA AND A. M. MANCHO, *The hidden geometry of ocean flows*, Physical Review Letters 105 (2010)
- [9] C. MENDOZA, A. M. MANCHO, AND M.-H. RIO, *The turnstile mechanism across the Kuroshio current: analysis of dynamics in altimeter velocity fields*, Nonlinear Proc. Geoph 17 (2010), 103–111.
- [10] J. A. MADRID AND A. M. MANCHO, *Distinguished trajectories in time dependent vector fields*, Chaos 19 (2009), 013111.
- [11] C. LOPESINO, F. BALIBREA, S. WIGGINS, A.M. MANCHO, *Lagrangian Descriptors for Two Dimensional, Area Preserving Autonomous and Nonautonomous Maps*, Communications in Nonlinear Science and Numerical Simulation 27 (2015), 40–51.
- [12] A. M. MANCHO, J. CURBELO, S. WIGGINS, V.J. GARCIA-GARRIDO, C. MENDOZA *Beautiful Geometries Underlying Ocean Nonlinear Processes* Chapter in the book. *A Voyage Through Scales*. Eds. Gnter Blschl, Hans Thybo, Hubert Savenije, Lois Lammerhuber. Publishers European Geophysical Union and Edition Lammerhuber (2015).
- [13] M.L. SMITH AND A.J. McDONALDS, *A quantitative measure of polar vortex strength using the function M* , J. Geophys Res: Atmos 119 (2014), 5966–5986.
- [14] E.L. REMPEL, A. CHIAN, A. BRANDENBURG, P. MUNOZ, AND S. SHADDEN, *Coherent structures and the saturation of a nonlinear dynamo*, J. Fluid Mech. 729 (2013), 309–329.
- [15] G. CRAVEN AND R. HERNANDEZ, *Lagrangian Descriptors of Thermalized Transition States on Time-Varying Energy Surfaces*, Physical Review Letters 115 (2015), 148301.
- [16] A. RUIZ-HERRERA, *Some examples related to the method of Lagrangian descriptors*, Chaos 25 (2015), 063112.
- [17] G. HALLER, *Lagrangian coherent structures*, Annual Review of Fluid Mechanics 47 (2015), 137–162.
- [18] E. AURELL, G. BOFFETTA, A. CRISANTI, G. PALADIN, AND A. VULPIANI, *Predictability in the large: an extension of the concept of Lyapunov exponent*, Journal of Physics A 30 (1997), 1.
- [19] M.R. ALLSHOUSE AND T. PEACOCK, *Refining finite-type Lyapunov exponents ridges and the challenges of classifying them*, Chaos 25 (2015), 087410.

- [20] G. FROYLAND, *An analytical framework for identifying finite-time coherent sets in time dependent dynamical systems*, Physica D 250 (2013), 1–19.
- [21] D. KARRASCH, F. HUHN, AND G. HALLER, *Automated detection of coherent Lagrangian vortices in two dimensional unsteady flows*, Proceedings of the Royal Society of London A 471 (2014), 20140639.
- [22] G. FROYLAND, *Dynamic isoperimetry and the geometry of Lagrangian Coherent Structures*, Nonlinearity 25 (2015), 087406.
- [23] T. PEACOCK AND G. HALLER, *Lagrangian Coherent Structures: The hidden skeleton of fluid flows*, Physics Today 66 (2013), 41.
- [24] G. BOFFETTA, G. LACORATA, G. REDAELLI, AND A. VULPIANI, *Detecting barriers to transport: A review of different techniques*, Physica D 159 (2001), 58–70.
- [25] G. LAPEYRE, *Characterization of finite-time Lyapunov exponents and vectors in two-dimensional turbulence*, Chaos 12 (2002), 688–698.